

Abstracts

Random Metric Graphs

Introductory talks

From Erdős–Rényi to geometry: an introduction to random graph models and their limits

Jan Hladky

Czech Academy of Sciences, Institute of Computer Science

Random graphs have been studied in combinatorics and probability for over sixty years, and the subject has developed its own catalogue of models, asymptotic regimes, and limit theories. This introductory talk surveys that landscape for an audience whose interests lie in spectral theory, stochastic geometry, and mathematical physics, with an emphasis on points of contact rather than completeness.

Starting from the classical Erdős–Rényi model, which serves throughout as a mean-field reference point against which the effect of geometry can be measured, I will move on to inhomogeneous models, in which the probability of an edge depends on latent attributes of its endpoints. I will then turn to limits of graph sequences and to what such limits say about the spectra of the associated operators, where the dense and sparse regimes behave rather differently and the sparse one connects to notions familiar from the theory of random operators.

No background in graph theory will be assumed. The choice of topics in the second half will be adapted to the interests of the audience.

Spectral Geometry of Graphs: an introduction

Pavel Kurasov

Stockholm University

The goal of this talk is to make informal but rigorous introduction into the theory of differential operators on metric graphs, their spectral theory and its interaction with the geometry/topology of the underlying graph.

Our focus will be on the Laplacian, but the developed methods can be applied to study Schrödinger, magnetic Schrödinger and any other differential operator.

Main topics to discuss:

- definition of a metric graph and their moduli spaces;
- elementary spectral theory for finite compact graphs;
- secular polynomials and secular functions;
- trace formula for graph Laplacian and its implications;
- surgery principles and spectral estimates;
- irreducibility of secular polynomials and arithmetic structure of the spectrum.

Networks, small-worlds, meanders: from spectral theory to geometry and back

Delio Mugnolo

FernUniversität in Hagen

This is a gentle introduction to the spectral theory of metric graphs, with a specific focus on spectral geometry: the study of links between the size and spacing of eigenvalues on the one hand, and geometric features of a physical structure on the other hand. I will especially stress the importance of a celebrated formula proved by J.P. Roth that relates the spectrum and cycle lengths, hence eigenvalues and tortuosity.

Stochastic microstructure modeling based on random graphs

Matthias Neumann

Graz University of Technology

Methods from spatial statistics and stochastic geometry have been shown to be powerful tools for realistically modeling micro- and nanostructures in functional materials such as batteries and solid oxide fuel cells. The stochastic models can then be used to generate large data sets of virtual yet realistic structures such that relationships between morphological descriptors of the considered structures and their corresponding macroscopic properties can be statistically investigated. In this talk, we give an overview about graph-based models, which are used to appropriately model connectivity properties in micro- and nanostructures. The presented examples include a germ-grain model for the collection of active material particles in battery electrodes as well as a generalized Voronoi-diagram, where the seed points are random graphs. The latter model has been used to generate virtual three-phase anode structures in solid-oxide fuel cells. Moreover, methods for fitting the parametric models to 3D image data of real structures are presented.

Random tessellations

Zbyněk Pawlas

Charles University Prague

Random tessellations are among the most extensively studied models in stochastic geometry. A tessellation of a d -dimensional domain is a finite or countable collection of compact sets with non-empty interiors, called cells, whose union covers the domain, whose interiors are pairwise disjoint, and such that every bounded subset intersects only finitely many cells. There are numerous approaches to constructing random tessellation models, each leading to different geometric and probabilistic properties. In this talk, we present several fundamental constructions of random tessellations and discuss their basic properties, illustrating the variety of models arising in stochastic geometry.

Benjamini–Schramm limit of the heat semigroup on quantum graphs

Eszter Sikolya

Eötvös Loránd University

We consider quantum graphs carrying second order operators (Hamiltonians) equipped with general boundary conditions in the vertices described by unitary matrices. Based on a paper of Anantharaman et al. from 2021, we define Benjamini–Schramm convergence of quantum graphs. For quantum graphs with uniformly bounded geometry, equipped with continuity and Kirchhoff vertex conditions, we show that the heat semigroup, transported to a common Hilbert space, depends continuously on the underlying rooted quantum graph with respect to a local Benjamini–Schramm- type metric. As a consequence, root-averaged pairings of the semigroup converge along Benjamini–Schramm convergent sequences of finite quantum graphs. The talk is about a joint work with Mihály Kovács.

Contributed talks

Sturmian discrete and metric graphs - appearance and absence of gaps

Ram Band

Technion

We study the spectra of Sturmian Schroedinger operators on graphs (either metric or discrete). A Sturmian sequence is an aperiodic sequence which is determined by an irrational frequency. These sequences are used to form aperiodic graph structures.

We consider the integrated density of states (IDS) for Schroedinger operators on such graphs. The gap labelling theory predicts all the possible gaps in the spectra of these operators. We explicitly compute the gap labels; we show which gaps appear in the spectrum and which are absent, and why.

The talk is based on joint works with Siegfried Beckus, Raphael Loewy and Gilad Sofer.

On the mobility edge in random quantum graphs

Pavel Exner

Czech Technical University & Czech Academy of Sciences

The aim of this short contribution is to turn attention to an open question concerning random quantum graphs. Molchanov and Safronov showed recently that a (hyper)cube lattice graph with random δ -coupling at the vertices has pure point spectrum at the edges of all spectral bands, that is, arbitrarily high in the spectrum. If you replace the basic cell of such a lattice by a (hyper)rectangle and choose the parameters appropriately, the present speaker together with Ondřej Turek showed that such a periodic graph can have a Bethe-Sommerfeld property, that is, a finite number of open gaps only, in particular, no gaps at all. The question naturally arises whether randomization of such a graph would have a mobility edge or not.

Geometry of the Wolfram Physics Project

Pavel Hajek

Wolfram Institute

Could simple machines generate enough complexity to explain the emergence of space-time geometry as an effective macroscopic description? Could hypergraph rewriting be a natural "machine code of the universe"? A concrete program was initiated by Stephen Wolfram, Max Piskunov, and Jonathan Gorard, focusing on building computational tools. I will outline the basic setting and definitions and speculate on potentially mathematically quantifiable phenomena and results

Hyperuniformity - measuring long-range order and disorder of random systems

Luca Lotz

DLR (German Aerospace Center)

We introduce hyperuniformity, a property of stationary point processes on $\mathbb{R}^d$ that describes the suppression of large-scale density fluctuations. We give some basic examples and then study when hyperuniformity is preserved under random displacements of the points. We illustrate our results with several examples, including isotropic hyperuniform point processes that can be simulated in linear time. In two and three dimensions, we can generate samples with 10^9 points. We also discuss a notion of hyperuniformity for random graphs embedded in $\mathbb{R}^d$ and give an example based on a modification of the Poisson-Delaunay graph.

Probing the influence of topological and geometric disorder on the spectrum of the differential Laplacian operator on networks

Charles Maher

New York University

Metric networks are network-shaped, one-dimensional structures on which one can solve differential equations to simulate a wide range of physical systems including conjugated molecules, photonic crystals, quantum mechanics in waveguide networks, and acoustic metamaterials. More concretely, a metric network is a network whose edges are each assigned a notion of length and a coordinate describing position. One can then define function spaces and differential operators on these objects to model the aforementioned systems. Recent software advancements have made it feasible to analyze partial differential equations on large, compact metric networks with a vast array of structures. Here, we generate compact metric network structures using the spatial tessellations of two-dimensional hyperuniform point patterns, which have suppressed large-scale density fluctuations relative to typical disordered point patterns. This choice of structure is inspired by the exotic physical properties of network materials with these structures in other contexts. Then, we characterize the eigenvalue spectrum structure of the differential Laplace operator on these networks. In particular, we find that gaps can form in the eigenvalue spectra of these networks whose widths increase when the distribution of edge lengths is narrow and as the number of triangular faces increases. Importantly, many of the structures we consider are realizable in Euclidean space, meaning they are well-suited for practical applications in, e.g., metamaterial design. This work can thus be used to inform the design of metric network-based systems with spectral gaps with tunable widths and locations.

The Brownian web as a random R -tree

Jan Swart

The Czech Academy of Sciences, Institute of Information Theory and Automation

The Brownian web can loosely be described as coalescing one-dimensional Brownian motions, started from every point in space and time. Its study has proved useful for a diverse collection of related models in theoretical probability. The Brownian web naturally gives rise to a random R -tree, as I will explain in this introductory talk.

Short talks

Planarity criteria for metric graphs

Alice Brolin

Stockholm University

The Colin de Verdière parameter is a number assigned to discrete graphs which equals the maximal multiplicity of the second eigenvalue of a certain family of Laplacian matrices related to the graph. We show that the Colin de Verdière parameter can be obtained in the setting of metric graphs by looking at the maximal multiplicity of the second eigenvalue for Laplacians on metric graphs with delta couplings at the vertices. Two different families of Laplacians, as well as a family of Schrödinger operators, all leading to the Colin de Verdière parameter are presented. This talk is based on joint work with Pavel Kurasov.

Hyperuniformity in the Bethe lattice

Mattias Byléhn

University of Vienna

Hyperuniformity of homogeneous random particle configurations in Euclidean geometries (and lattices) is characterized by the variance of the number of particles in large domains being asymptotically suppressed by that of an invariant Poisson point process/ideal gas. Equivalently, hyperuniformity can be detected from the suppression of long wavelengths in the diffraction picture of the configuration. In this talk we will see how hyperuniformity extends to the setting of automorphism-invariant random configurations in the Bethe lattice/regular trees with emphasis on examples.

Complexity of a Randomly Initialized Neural Network Through Its Affine Regions: A Higher-Dimensional Random-Polytope Viewpoint and the Possible Role of Spectral Properties of Large Weight Matrices

Recep Özkan

Aarhus University & Middle East Technical University (METU)

We study Gaussian ReLU networks using a conditional Kac–Rice method: conditioning on the preceding layers, each new preactivation is a Gaussian field, affine on the cells of a random partition inherited from those layers, which gives an exact formula for the expected measure of its zero set at finite width. As the widths grow proportionally, the conditional covariances converge to deterministic recursions across layers, yielding an explicit layerwise formula for the expected measure of the *scalar kink set*: the set where the network output fails to be differentiable. For $d \geq 2$ this is the leading $(d - 1)$ -dimensional Hausdorff measure, via a cellwise coarea formula and a Crofton representation using random lines; for $d = 1$ it specializes to an explicit zero-crossing count.

For $d \geq 2$ this measure result has a random-polytope viewpoint: the affine regions are the cells of a random hyperplane-type arrangement, each a random polytope, whose boundaries form a random metric graph. Counting affine regions is therefore a question about this arrangement's vertices, edges, and faces, not only its total boundary measure — a count that higher-order Kac–Rice formulas, combined with the combinatorics and topology of the resulting cell complex, may supply, though our present first-order theory alone cannot. The model is currently rotationally isotropic; replacing the isotropic weight covariance by one with a nontrivial spectrum, such as a limiting spectrum from random matrix theory, would let that spectrum act on the local geometry of the arrangement through the same recursion — the possible role of weight-matrix spectra we flag.

We flag two directions here. The first is the second-order computation above. The second inverts the construction, letting the input domain itself be a metric graph, such as the edge skeleton of a Poisson–Voronoi or Poisson–Delaunay tessellation, or its hyperuniform analogue; along each edge the one-dimensional theory applies without change, while a vertex of degree $k \geq 3$ introduces a new kind of singularity whose treatment is a modeling choice, and the graph's topology enters the region count as a plain interval never does. The random graphs arising this way are new to this setting, and warrant further study as domains for Laplace-type operators.

This is joint work with Christian Hirsch.

The discrete Steklov problem

Miloš Puđa

IMFM & University of Ljubljana

For a nice enough manifold with boundary, the Dirichlet-to-Neumann operator takes a function defined on the boundary and maps it to the normal derivative of its unique harmonic extension into the interior. The Steklov problem involves determining the spectrum of this operator. We will discuss a discrete version of this problem for graphs (with boundary) and talk about some general questions and directions regarding the topic.

Random Laguerre Tessellation

Martina Švarc Petráková

Charles University Prague

Among the prominent topics of stochastic geometry lies the study of random tessellations. One of the classical families is formed by random Voronoi tessellation, its generalizations and their dual tessellations. In this talk, we focus on the weighted generalization of random Laguerre tessellation, which arises by partitioning the Euclidean space based on a power distance to a random set of weighted points. We discuss several properties of this model, focusing mostly on the case of Poisson-Laguerre tessellation, where the random set of generators is an independently marked Poisson point process (i.e. the case with complete spatial randomness).

